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Endomorphisms of the Separated Product of Lattices

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To describe the evolution of separated entities remaining separated, we propose to study endomorphisms (join-preserving maps, sending atoms to atoms) of the separated product of cao lattices (complete, atomistic orthocomplemented lattices). Morphisms have been used successfully to describe the evolution of entities, and the separated product is a model for the property lattice of separated systems; its set of atoms is the Cartesian product of each atom space. Let $\mathscr{L}$ be the separated product of two cao lattices having the covering property and f an endomorphism of $\mathscr{L}$. We prove that the center $\mathscr{Z}(\mathscr{L})$ of $\mathscr{L}$ is the power set of $\Omega_1 \times \Omega_2$ where Ω_i is the atom space of $\mathscr{Z}(\mathscr{L}_i)$ (Theorem 1), f preserves irreducible components (Theorem 2), and if $\mathscr{L}$ is irreducible there exist two endomorphisms f^1 and f^2 and a permutation σ such that the restriction of f to atoms is given by $f(p_1, p_2) = (f^1(p_{\sigma(1)}), f^2(p_{\sigma(2)}))$ (Theorem 3). For generalizations of these results to separated products of families of cao lattices, we develop new general arguments involving a topology we define on the set of atoms of a cao lattice.

1. INTRODUCTION AND NOTATIONS

We first recall some notions used in the foundation of physics to describe physical systems. For details, see Piron (1976), Aerts (1982), and Faure *et al.* (1995).

A question α on a physical system S is an experiment with two outcomes called "yes" and "no." The inverse $\alpha^{\sim}$ of a question α is the question obtained by inverting the answers "yes" and "no." The product $\Pi_{i\in I}\, \alpha_i$ of a family of questions $\{\alpha_i\}_{i\in I}$ is the question which consists in choosing one question α_i and to attribute to $\Pi_{i\in I}\, \alpha_i$ the answer thus obtained. The trivial question denoted I consists in measuring anything (or doing nothing) and answering always "yes." A question α is said to be certain if when a physicist decides to perform it, the answer "yes" comes out with certainty. Finally, a question

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α is said to be stronger than a question β if we have that "α certain" implies "β certain". This preorder relation induces an equivalence relation. To each equivalence class $[\alpha]$ there corresponds a property a of the system which is said to be actual if α is certain. The set $\mathcal{L}$ of equivalence classes is a complete lattice, the greatest lower bound of a family $\{a_i\}_{i\in I}$ being the class defined by the question $\Pi_{i\in I}\,\alpha_i$ and the maximal element being the class defined by the trivial question I. The lattice $\mathcal{L}$ is called the property lattice. The state of the system is the set of all actual properties. Two states ε and ε' are said to be orthogonal, $\varepsilon \perp \varepsilon'$, if there exists a question α such that α is certain when the system is in the state ε and $\alpha^{\sim}$ is certain when the system is in the state ε'. For a given state ε, write $p_\varepsilon := \wedge\{a \in \mathcal{L};\ a \in \varepsilon\}$; then, by definition, $p_\varepsilon \in \varepsilon$ and $\varepsilon = [p_\varepsilon, I] := \{a \in \mathcal{L};\ p_\varepsilon < a\}$. If p is an atom of $\mathcal{L}$, then p is actual if and only if the state of the system is $\varepsilon_p = [p, I]$. If one supposes that for each state ε of the system, (i) there exists a question α_ε such that α_ε is true if and only if the state of the system is orthogonal to ε, (ii) p_ε is an atom of $\mathcal{L}$, and (iii) every question on the physical system is equivalent to a product of primitive questions, then $\mathcal{L}$ is atomistic and orthocomplemented by $a' = \wedge\{[\alpha_\varepsilon];\ p_\varepsilon < a\}$, and $\mathcal{L}$ is orthoisomorphic[2] to $\mathcal{C}(\Sigma, \perp)$, the lattice of biorthogonal subsets ($A^{\perp\perp} = A$) of the set Σ of possible states.

Further, a property a is called classical if for any state ε, either a is actual or a' is actual. The set $\mathcal{Z}(\mathcal{L})$ of all classical properties of $\mathcal{L}$ is a subcomplete, atomistic orthocomplemented (cao) lattice. The system is said to be purely quantum if and only if $\mathcal{Z}(\mathcal{L}) = \{O, I\}$. We will see later that $\mathcal{Z}(\mathcal{L})$ is the center of $\mathcal{L}$ and so $\mathcal{L}$ is purely quantum if and only if $\mathcal{L}$ is irreducible. Finally $\mathcal{L}$ is orthoisomorphic to the Cartesian product of the irreducible components of $\mathcal{L}$.

To describe evolution, one has to remark that a given evolution is nothing more than part of an experimental project. Let Q_t denote the family of all questions which can be performed on the system at a time t. Consider a question $\alpha \in Q_{t_1}$ and denote by $\Phi_{01}\,(\alpha)$ the question defined by "evolve the system from time t_0 to time t_1 and perform α." So, by definition of Q_{t_0}, $\Phi_{01}(\alpha) \in Q_{t_0}$ and Φ_{01} defines an application from Q_{t_1} to Q_{t_0}. One can check that Φ_{01} preserves the product and the equivalence relation, so that Φ_{01} defines a map $\phi_{01}\colon \mathcal{L}_{t_1} \to \mathcal{L}_{t_0}$ which preserves the meet and sends O_{t_1} to O_{t_0}. If for a certain state p of the system at time t_0, the system may disappear during the evolution, then $p \wedge \phi_{01}\,(I_{t_1}) = O_{t_0}$ since for I_{t_1} to be certain, the system has to exist. Let $p_0 < \phi_{01}\,(I_{t_1})$ be the state of the system at time t_0. Define $\psi_{10}(p_0)$ as the smallest actual property of the system at

[2] We call an orthoisomorphism a bijection between two cao lattices which preserves the join, the meet, and the orthocomplementation.

time t_1, that is, $\psi_{10}(p_0) = \wedge\{a_1 \in \mathscr{L}_{t_1}; p_0 < \phi_{01}(a_1)\}$. If for every state $p_0 < \phi_{01}(I_{t_1})$, $\psi_{10}(p_0)$ is an atom of $\mathscr{L}_{t_1}$, then the final state of the system is completely determined and the evolution is said to be deterministic. Moreover, the application ψ_{10}: $[O_{t_0}, \phi_{01}(I_{t_1})] \to \mathscr{L}_{t_1}$ preserves the join and sends atoms to atoms and O_{t_0} to O_{t_1}. We will call such a map a morphism. If ψ_{10} preserves irreducible components, then the evolution of classical variables does not depend on the actual quantum state. So, for a model of property lattice and evolution given by a cao lattice and a morphism, it is important for the physical interpretation that the morphism preserve irreducible components. It is clear that not any endomorphism of a cao lattice preserves irreducible components, as we will see on an example in Section 3.

We now briefly recall the construction of the model proposed by Aerts for the property lattice of a physical system S constituted of two separated subsystems S_1 and S_2.

Two questions α and β are said to be testable together if and only if there exists an experimental project $E(\alpha, \beta)$ with four outcomes, labeled by *yy*, *yn*, *ny*, and *nn*, such that $\alpha \sim E_{yy,yn}$, $\alpha^{\sim} \sim E_{ny,nn}$, $\beta \sim E_{yy,ny}$, and $\beta^{\sim} \sim E_{yn,nn}$, where, for example, $E_{yy,yn}$ is the question consisting in performing $E(\alpha, \beta)$ and answering "yes" if the result is *yy* or *yn* and "no" otherwise. Two questions α and β which are testable together are said to be separated if and only if, when for an arbitrary state p of the system there is a certain chance to obtain an answer for α and another for β, then for this state of the system there is a certain chance to obtain this combination for $E(\alpha, \beta)$. Finally, the systems S_1 and S_2 are said to be separated if and only if every question of S_1 is separated from every question of S_2. The model of Aerts is given by a set Q of questions on all the system S: Q is the union of Q_1, Q_2, and all the questions of the form $E(\alpha_1, \alpha_2)_{.,.}$ where $\alpha_1 \in Q_1$ and $\alpha_2 \in Q_2$, closed relative to the product of questions. From this, one can prove that the set of states of S is given by $\Sigma = \Sigma_1 \times \Sigma_2$ and that the orthogonality relation between states is given by

$$(\varepsilon_1, \varepsilon_2) \perp_{\oslash} (\eta_1, \eta_2) \Leftrightarrow \varepsilon_1 \perp \eta_1 \text{ or } \varepsilon_2 \perp \eta_2 \tag{1}$$

(Aerts, 1982, Theorems 19 and 21). One can also prove that if S_1 and S_2 satisfy the basic axioms mentioned above, then S described by Q also satisfies these axioms, so that the property lattice $\mathscr{L}$ of S is cao and

$$\mathscr{L} \simeq \mathscr{L}_1 \oslash \mathscr{L}_2 := \mathscr{C}(\Sigma_1 \times \Sigma_2, \perp_{\oslash}) \tag{2}$$

We can now turn to the mathematical part of this paper. For this purpose, we need to introduce some convenient notations and to recall some results about cao lattices.

Let $\mathscr{L}$ be a cao lattice. We denote Σ the set of atoms of $\mathscr{L}$, and $\perp$ the binary relation on Σ induced by the orthocomplementation of $\mathscr{L}$: $p \perp q \Leftrightarrow$

$p < q'$. Then $\mathscr{L}$ is orthoisomorphic to $\mathscr{C}(\Sigma, \perp)$ and $\perp$ is an orthogonality relation, that is, $\perp$ is symmetric, antireflexive, and separating, i.e., for every pair p, q of atoms of $\mathscr{L}$, there exists an atom k such that $p \perp k$ and $q \not\perp k$ or equivalently, every singleton of Σ is biorthogonal. Reciprocally, if $\perp$ is an orthogonality relation on a set Σ, then $\mathscr{C}(\Sigma, \perp)$ is a cao lattice. For two elements a and b of a cao lattice, we will write $a \vee b = a \cup b$ if and only if every atom under $a \vee b$ is under a or under b. Finally, we denote by $\mathscr{P}(\Omega)$ the cao lattice of subsets of Ω.

We will denote by $\mathscr{Z}(\mathscr{L})$ the sub cao lattice of elements of $\mathscr{L}$ such that $a^\perp = a^c$, the complementary set of a in Σ, and by Ω the set of atoms of $\mathscr{Z}(\mathscr{L})$. It is easy to verify that for every family $\{a_i\}_{i \in I}$ of elements of $\mathscr{L}$ and for every $\alpha \in \mathscr{Z}(\mathscr{L})$ we have that $a_i = (a_i \wedge \alpha) \vee (a_i \wedge \alpha^\perp)$, $a_i = \vee_{\alpha \in \Omega}(a_i \wedge \alpha)$ and $(\vee_{i \in I}\, a_i) \wedge \beta = \vee_{i \in I}\,(a_i \wedge \beta)$. From these relations it follows that $\mathscr{L}$ is orthoisomorphic to the Cartesian product $\Pi_{\alpha \in \Omega}\,[O, \alpha]$, where $[O, \alpha] = \{a \in \mathscr{L}; a < \alpha\}$ and the orthocomplementation is the induced orthocomplementation $a^r = a^\perp \wedge \alpha$ $(a^{rr} = ((a^\perp \wedge \alpha) \vee \alpha^c)^\perp = a)$. $\mathscr{Z}(\mathscr{L})$ is the center of $\mathscr{L}$: Indeed, let $\alpha \in \mathscr{Z}(\mathscr{L})$; then $\mathscr{L} \simeq [O, \alpha] \times [O, \alpha^\perp]$, and reciprocally, let $a \in \mathscr{L}$, and $\mathscr{L}_1$ and $\mathscr{L}_2$ be two cao lattices such that $\mathscr{L} \simeq \mathscr{L}_1 \times \mathscr{L}_2$ and a corresponds to (I_1, O_2); then $a^\perp$ corresponds to (O_1, I_2); so $a^\perp = a^c$. In consequence, purely quantum corresponds to irreducible. For an atom p of $\mathscr{L}$, we denote by $\mathscr{Z}(p)$ the central cover of p, that is, $\mathscr{Z}(p) = \wedge\{\alpha \in \Omega; p < \alpha\}$.

For a cao lattice having the covering property, we have the following property we will use frequently: let p and q be two atoms of $\mathscr{L}$; then $\mathscr{Z}(p) = \mathscr{Z}(q) \Leftrightarrow p \vee q \neq p \cup q$. With the relations we gave before, it is easy to see that if $\mathscr{Z}(p) \neq \mathscr{Z}(q)$, then $p \vee q = p \cup q$. To show the converse, we first have to remark that in a cao lattice having the covering property, we have that $p \vee q = p \cup q \Rightarrow p \perp q$ or $p = q$. Indeed, if $p \vee q = p \cup q$, then $x := (p \vee q) \wedge q^\perp = p \wedge q^\perp$ and so $O < x < p$. If $x = p$, then $p < q^\perp$ and if $x = O$, then $[p^\perp \wedge q^\perp] \vee q = I$, that is, I covers $p^\perp \wedge q^\perp$, so $p = q$. In consequence, it remains to show that if $\mathscr{Z}(p) = \mathscr{Z}(q)$ and $p \perp q$, then $p \vee q \neq p \cup q$. We write $p \sim q$ if there exist a finite number of atoms $z_1, \ldots, z_n$ such that $p \not\perp z_1$, $z_i \not\perp z_{i+1}$, and $z_n \not\perp q$. We call note $[p]$ the equivalence class of p; then $[p] \in \mathscr{Z}(\mathscr{L})$ and $[p] = \mathscr{Z}(p)$ because if $q \notin [p]$, then $q \in [p]^\perp$. Finally, the relation $p \vee q \neq p \cup q$ is transitive: Indeed, consider three atoms p, z, and q such that there exist r_1 different from p and z, and r_2 different from z and q, with $r_1 < p \vee z$ and $r_2 < z \vee q$. Write $a = r_1 \vee r_2$ and $x = (p \vee q) \wedge a$. By the covering property, $p < q \vee a$, which implies that $x \neq O$. Moreover, if $x = a$, then $z < p \vee q$, otherwise x is an atom different from p and q.

Finally, for any irreducible cao lattice having the covering property of rank greater than or equal to 4, there exist a vector space V on a field $\mathbb{K}$, an involution σ on $\mathbb{K}$, and a Hermitian form ϕ on V such that $\mathcal{L} \simeq \mathcal{C}(V/\mathbb{K}^*, \perp)$ where $\perp$ is induced by ϕ (Maeda and Maeda, 1970; Piron, 1964). If f is a nontrivial endomorphism of $\mathcal{L}$, that is, a join-preserving map, sending atoms to atoms such that the rank of $\mathrm{Im}(f)$ is greater than or equal to 3, then f is induced by a semilinear map g on V (Faure and Frölicher, 1993, Definition 4.1.1 and Proposition 4.1.2; Faure and Frölicher, 1994, Theorem 5.1.5). Moreover, if $\mathcal{L}$ is orthomodular, if $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and if the involution is the usual one, then V is a Hilbert space. In this case, if g is quasilinear and if $f(p) \perp f(q)$ implies $p \perp q$ for any atoms p and q, then g is unitary or antiunitary (Faure *et al.*, 1995, Theorem 4.2).

2. THE SEPARATED PRODUCT

The separated product can be defined in the same way as in (1) and (2) for an arbitrary family $\{\mathcal{L}_\alpha\}_{\alpha\in\omega}$ of cao lattices by

$$\bigotimes_{\alpha\in\omega} \mathcal{L}_\alpha = \mathcal{C}\left(\prod_{\alpha\in\omega} \Sigma_\alpha,\ \perp_{\otimes}\right) \tag{3}$$

where

$$p \perp_{\otimes} q \Leftrightarrow \exists\ \alpha \in \omega \text{ such that } p_\alpha \perp_\alpha q_\alpha$$

It is easy to check that $\perp_{\otimes}$ is separating, and moreover an orthogonality relation. In consequence, $\bigotimes_{\alpha\in\omega} \mathcal{L}_\alpha$ is also a cao lattice.

The only result concerning the separated product is due to Aerts: if the separated product $\bigotimes \mathcal{L}_\alpha$ has the covering property or if it is orthomodular, then at most one $\mathcal{L}_\alpha$ is not equal to its center (Aerts, 1982, Theorem 30, for the case $\#\omega = 2$). To prove this, one first has to remark that in a cao lattice $\mathcal{L}$ having the covering property or which is orthomodular, we have the following relation: $p \vee q = p \cup q \Rightarrow p \perp q$ or $p = q$. We have already proved this when $\mathcal{L}$ has the covering property. If $\mathcal{L}$ is orthomodular, then $[(p \vee q) \wedge q^\perp] \vee q = p \vee q$, so that if $x := (p \vee q) \wedge q^\perp = O$, then $p \vee q = q$.

Finally, if, for example $\mathcal{L}_\alpha \neq \mathcal{Z}(\mathcal{L}_\alpha)$, then there exist two atoms p and q of $\mathcal{L}_\alpha$ such that $p \not\perp_\alpha q$. Let $\beta \neq \alpha$, and let r, s be two atoms of $\mathcal{L}_\beta$. Let x and y be atoms of the separated product such that $x_\alpha = p$, $x_\beta = r$, $y_\alpha = q$, $y_\beta = s$, and $x_\gamma = y_\gamma$ for $\alpha \neq \gamma \neq \beta$. Then $p \vee_{\otimes} q = p \cup q$ (Lemma 1.b) and so, if the separated product has the covering property or if it is orthomodular, $p \perp_{\otimes} q$, that is, $r \perp_\beta s$, which shows that $\mathcal{Z}(\mathcal{L}_\beta) = \mathcal{L}_\beta$.

The results that follow are original; they cannot be founded in Aerts (1982). Before proving the theorems announced in the abstract, we need two preliminary lemmas and a definition we will use throughout the paper:

Definition 1. Let $\{\mathscr{L}_\alpha\}_{\alpha\in\omega}$ be a family of cao lattices. (a) For any nonempty subset B_α of Σ_α, we define $o(B_\alpha) \subset \Pi_{\alpha\in\omega}\,\Sigma_\alpha$ by $o(B_\alpha)_\beta = \Sigma_\beta$ for $\beta \neq \alpha$ and $o(B_\alpha)_\alpha = B_\alpha$. Moreover, we put $o(\emptyset) = \emptyset$.

(b) Let p and q be two atoms of the separated product. We define $C(p, q) \subset \Pi_{\alpha\in\omega}\,\Sigma_\alpha$ by $C(p, q)_\alpha = p_\alpha \vee_\alpha q_\alpha$.

From now on, we will drop the subscripts $\oslash$ and α when no confusion can occur.

Lemma 1. Let $\{\mathscr{L}_\alpha\}_{\alpha\in\omega}$ be a family of cao lattices and a and b elements of $\Pi_{\alpha\in\omega}\,\mathscr{L}_\alpha$ such that $a_\alpha \neq O \neq b_\alpha$, $\forall\alpha \in \omega$. We have:

(a) $a \in \bigoslash_{\alpha\in\omega} \mathscr{L}_\alpha$.

(b) $a \vee b = a \cup b \cup_{\alpha\in\omega} y^\alpha$, where $y^\alpha_\alpha = a_\alpha \vee b_\alpha$ and $y^\alpha_\beta = a_\beta \wedge b_\beta$ for $\beta \neq \alpha$.

(c) We note $x^\alpha_\alpha = a_\alpha$ and $x^\alpha_\beta = b_\beta$ for $\beta \neq \alpha$. Suppose that $a_\alpha < b_\alpha$, $\forall\alpha \in \omega$. Then $\vee_{\alpha\in\omega}\, x^\alpha = \cup_{\alpha\in\omega} x^\alpha$.

(d) Suppose that ω is finite. Let $x^1 = a$, $x^2_1 = b_1$, and $x^2_i = a_i$ for $i \geq 2$, $x^k_i = b_i$ if $i \leq k - 1$ and $x^k_i = a_i$ if $i \geq k$. Then $\vee_{k\in\omega}\, x^k = \cup_{k\in\omega}\, z^k$, where $z^k_k = a_k \vee b_k$, $z^k_i = a_i$ if $i \geq k + 1$ and $z^k_i = b_i$ if $i \leq k - 1$.

(e) Let $\{p^\beta;\ \beta \in \omega\}$ be a set of atoms of $\oslash_{\alpha\in\omega}\,\mathscr{L}_\alpha$ such that $p^\beta_\alpha \neq p^\gamma_\alpha$, $\forall\alpha, \beta, \gamma \in \omega$. Then $\vee_{\beta\in\omega}\, p^\beta = \cup_{\beta\in\omega} p^\beta$.

(f) Let p be an atom of $\oslash_{\alpha\in\omega}\,\mathscr{L}_\alpha$ and $\{q^\beta;\ \beta \in \omega\}$ be a set of atoms such that for every β, $\#\{\alpha;\ q^\beta_\alpha \neq p_\alpha\} = \#\omega$. Then $p \wedge (\vee_{\beta\in\omega}\, q^\beta) = O$.

Proof. (a) By definition (3) of $\perp_\oslash$, we have that $a^\perp = \cup_{\alpha\in\omega}\, o(a^\perp_\alpha)$. So,

$$a^{\perp\perp} = \left(\bigcup_{\alpha\in\omega} o(a^\perp_\alpha)\right)^\perp = \bigcap_{\alpha\in\omega} o(a^{\perp\perp}_\alpha) = a$$

(b) Let us first consider a set $X \subset \Pi_{\alpha\in\omega}\,\mathscr{L}_\alpha$. Then, in the same way as before, we find

$$\begin{aligned}\bigvee X &= \left\{\bigcap_{x\in X}\bigcup_{\alpha\in\omega} o({x_\alpha}^\perp)\right\}^\perp = \left\{\bigcup_{h\in\omega^X}\bigcap_{x\in X} o(x^\perp_{h(x)})\right\}^\perp \\ &= \bigcap_{h\in\omega^X}\ \bigcup_{\alpha\in\mathrm{Im}(h)} o\left(\bigvee_{x\in h^{-1}(\alpha)} x_\alpha\right) \\ &= \bigcup_{\xi:\omega^X\to\mathrm{Im}(\cdot)}\ \bigcap_{h\in\omega^X} o(\bigvee\{h^{-1}(\xi(h))_{\xi(h)}\}) \\ &= \bigcup_{\xi:\omega^X\to\mathrm{Im}(\cdot)} x^\xi \end{aligned} \tag{4}$$

For part (b), we have $X = \{a, b\}$. As a consequence, if h is a constant function, then $h^{-1}(\xi(h))_{\xi(h)} = \{a_{\xi(h)}, b_{\xi(h)}\}$ and otherwise $h^{-1}(\xi(h))_{\xi(h)}$ is equal to $a_{\xi(h)}$ or $b_{\xi(h)}$. Thus, for any $\alpha \in \omega$, x^{ξ}_{α} is equal to a_{α}, b_{α}, $a_{\alpha} \wedge b_{\alpha}$, or $a_{\alpha} \vee b_{\alpha}$.

If, for example, $x^{\xi}_{\alpha} = a_{\alpha}$ for a certain α, then $x^{\xi}_{\beta} < a_{\beta}$, $\forall \beta \neq \alpha$, since if $x^{\xi}_{\alpha} = a_{\alpha}$, $\xi(h) = h(a)$ for every function $h \in \omega^X$, such that $h(b) = \alpha$.

Finally, if $x^{\xi}_{\alpha} = a_{\alpha} \vee b_{\alpha}$, then $x^{\xi}_{\beta} = a_{\beta} \wedge b_{\beta}$ because $\xi(h) \neq \alpha$ for every nonconstant function whose image contains α.

(c) In this case, $X = \{x^{\alpha}; \alpha \in \omega\}$. Because of the hypothesis $a_{\alpha} < b_{\alpha}$, $\vee\{h^{-1}(\xi(h))_{\xi(h)}\} = b_{\alpha}$ for every constant function h. In consequence, $x^{\xi} < b$, $\forall \xi$. Moreover, since ω^X contains the identity, there exists α such that $x^{\xi} < x^{\alpha}$.

(d) In this case, $X = \cup_{k \in \omega} x^k$. Let $h \in \omega^{\omega}$ be a step constant function, that is, for any $k \in \mathrm{Im}(h)$, $h^{-1}(k)_k$ is equal to a_k or b_k. Denote k_a those k in $\mathrm{Im}(h)$ such that $h^{-1}(k)_k = a_k$ and similary k_b. If $\mathrm{Im}(h)$ contains no k_b, then $\mathrm{Im}(h)$ is not bounded. Otherwise, there exist k_b and k_a such that $k_b < k_a$.

Let $i \in \omega$ and h be a step constant function. If there exists $k_b < i$ in $\mathrm{Im}(h)$, put $\xi(h) = k_b$, otherwise put $\xi(h) = k_a$ with $k_a > i$. For any other function h, choose $\xi(h)$ such that $\vee\{h^{-1}(\xi(h))_{\xi(h)}\} = a_{\xi(h)} \vee b_{\xi(h)}$. Then $x^{\xi} = z^i$.

Finally, let i be the smallest element in ω such that $x^{\xi}_i \not< b_i$. Then $\xi(h) = h(1)$ for every function h of the type $h(x^k) = i$, $\forall\, k > i$ and $h(x^k) = j > i$, $\forall k \leq i$. As a result, $x^{\xi}_k < a_k$, $\forall k > i$, and so $x^{\xi} < z^i$.

(e) This result follows directly from the fact that for any nonconstant function $g \in \omega^{\omega}$, there exists a bijection h^{-1} of ω such that $g(\alpha) \neq h^{-1}(\alpha)$, $\forall\, \alpha \in \omega$.

(f) If x^{ξ} is an atom, then $x^{\xi} \neq p$ since by hypothesis, there exists injective functions $h \in \omega^{\omega}$ such that $q^{\beta}_{h(\beta)} \neq p_{h(\beta)}$, $\forall \beta \in \omega$. Thus, for p to be in $\vee_{\beta \in \omega} q^{\beta}$, there must exist ξ such that x^{ξ}_{α} is of rank at least 2 for a set R of α such that $\#R = \#\omega$. This is impossible because ω^{ω} contains injective functions from ω to R. ■

Part (c) of Lemma 1 will be used in the proof of Theorem 3, part (d) in Remark 2, and parts (e) and (f) in Remark 1.

Lemma 2. Let $\{\mathcal{L}_{\alpha}\}_{\alpha \in \omega}$ be a family of cao lattices having the covering property. Denote by τ the topology on $\Pi_{\alpha \in \omega} \Sigma_{\alpha}$ which admits the biorthogonal subsets as a subbasis of closed sets. Let p and q be two atoms of $\oslash_{\alpha \in \omega} \mathcal{L}_{\alpha}$. Suppose that for any α, $C(p, q)_{\alpha}$ contains either only one atom or an infinity. Then $C(p, q)$ is a connected set for the topology τ.

Proof. Let f be a closed set of τ. Then, by definition of τ, $f = \cap_{\gamma} \cup_{j=1}^{n_{\gamma}} X^{j\gamma}$, where $X^{j\gamma} \in \oslash_{\alpha \in \omega} \mathcal{L}_{\alpha}$.

We write C for $C(p, q)$. The lemma is a direct consequence of the following remark: for any biorthogonal $X < C$ different from C, their exists an atom $p < C$ such that $X < \cup_{\alpha \in \omega} o(p_{\alpha}) \wedge C$. Indeed, we know that X is of the form (4) and since every $\mathcal{L}_\alpha$ has the covering property, $\vee\{h^{-1}(\alpha)_\alpha\}$ is an atom or C_α. If there exists no step constant function in ω^X, that is, if for any h, there exists α such that $\vee\{h^{-1}(\alpha)_\alpha\} = C_\alpha$, then $X = C$. Let h be a step constant function. Then $x^\xi < o(\vee\xi(h))$, $\forall\xi$.

Thus, for any closed sets f and f' of τ, different from C and $\emptyset$, there exists a finite set R of atoms of C such that $C\backslash R \subset (f^c \cap f'^c) \cap C$. ■

Remark 1. Suppose that ω is finite (write $I = \omega$) and that for any $k \in \omega$ and any pair r_k, s_k of atoms of C_k there exists an injective or constant function from [0, 1] to C_k such that $c_k(0) = r_k$ and $c_k(1) = s_k$. Then C is connected by arcs. Indeed, let r and s be two atoms of C. Define c from [0, 1] to C by $c(t) = (c_1(t), c_2(t), \ldots)$. Let x be a biorthogonal subset of C. Since I is finite, x contains at most a finite set of atoms $\{p^k; k \in I\}$ such that $p^j_k \neq p^l_k$, $\forall_j \neq l$ and k, otherwise, there is no step constant function in I^I and $x = C$. As a consequence, $c^{-1}(x)$ is a finite set of points, which shows, by definition of τ, that c is continuous.

But if ω is infinite and C_α is not an atom $\forall$ α, then C is not connected by arcs. First, remark that the argument for finite ω does not work in this case because for every noncompact converging sequence t_n, c^{-1} $(\vee_n c(t_n))$ is not closed since by Lemma 1(e), $\vee_n c(t_n) = \cup_n c(t_n)$.

Consider two atoms r and s of C such that $r_\alpha \neq s_\alpha$ $\forall$ α. Let c: [0, 1] $\rightarrow$ C with $c(0) = r$ and $c(1) = s$ and $\Omega := \{t \in [0, 1]; \#\{\alpha; c(t)_\alpha \neq s_\alpha\} \neq \#\omega\}$. Let t_0 be the greatest lower bound of Ω. If $t_0 \notin \Omega$, consider a sequence $t_n \rightarrow t_0$ with $t_n \in \Omega$. Then $c^{-1}(\vee_n c(t_n))$ is not closed, since by Lemma 1(f), $\vee_n c(t_n) \wedge c(t_0) = O$. If $t_0 \in \Omega$, consider a sequence $t_n \rightarrow t_0$ with $t_n < t_0$. Thus, c is not continuous.

We now state or first result concerning the center of the separated product. For the proof of part (b) of Theorem 1, we develop a new general argument involving the topology defined in Lemma 2.

Theorem 1. Let $\{\mathcal{L}_\alpha\}_{\alpha\in\omega}$ be a family of cao lattices.

(a) Suppose that ω is finite. Then the separated product $\oslash_{i\leq n} \mathcal{L}_i$ is irreducible if and only if $\mathcal{L}_i$ is irreducible for all i. Moreover, we have that $\mathfrak{X}(\oslash_{i\leq n} \mathcal{L}_i) = \mathcal{P}(\Pi_{i\leq n} \Omega_i)$.

(b) Suppose that for any $\alpha \in \omega$, $\mathcal{L}_\alpha$ has the covering property and the join of every pair of atoms of $\mathcal{L}_\alpha$ having the same central cover contains an infinity of atoms. Then the result of part (a) is true for an arbitrary family.

Proof. (a) This proof is only based on the definition (1) of $\perp_\oslash$. We first prove that $\mathcal{L}_1 \oslash \mathcal{L}_2$ is irreducible if and only if $\mathcal{L}_1$ and $\mathcal{L}_2$ are irreducible. Since the separated product is associative, the same result is true for any finite family of cao lattices.

First, remark that a cao lattice is irreducible if and only if for any nonempty subset $A \subset \Sigma$ different from Σ, $A^\perp$ is strictly include in A^c, the complementary set of A, that is, if and only if $A^{\perp c} \cap A^c \neq \emptyset$. Indeed, if $A^{\perp} = A^c$, then $A \subset A^{\perp\perp} = A^{c\perp} \subset A^{cc} = A$, so $A = A^{\perp\perp}$ and $A^{\perp} = A^c$, which is impossible in an irreducible cao lattice.

Suppose, for example, that $\mathscr{L}_1$ is reducible; then there exists $A \subset \Sigma_1$ such that $A^{\perp} = A^c = \Sigma_1 \backslash A$. So, by definition, $(A, \Sigma_2)^{\perp} = (A^{\perp}, \Sigma_2) = (A^c, \Sigma_2) = \Sigma_1 \times \Sigma_2 \backslash (A, \Sigma_2)$, which is impossible if $\mathscr{L}_1 \oslash \mathscr{L}_2$ is irreducible.

Suppose now that $\mathscr{L}_1$ and $\mathscr{L}_2$ are irreducible. Let $A \subset \Sigma_1 \times \Sigma_2$ a nonempty set different from $\Sigma_1 \times \Sigma_2$; then we have to show that $X := A^c \cap A^{\perp c} \neq \emptyset$. By definition,

$$A^{\perp} = \bigcap_{(p,q)\in A} [(p^{\perp}, \Sigma_2) \cup (\Sigma_1, q^{\perp})]$$

$$A^{\perp c} = \bigcup_{(p,q)\in A} [(p^{\perp c}, \Sigma_2) \cap (\Sigma_1, q^{\perp c})] = \bigcup_{(p,q)\in A} (p^{\perp c}, q^{\perp c})$$

For a subset $S \subset A$, write $S_1 := \{p \in \Sigma_1 | (p, \Sigma_2) \cap S \neq \emptyset\}$ and $S_2 := \{q \in \Sigma_2 | (\Sigma_1, q) \cap S \neq \emptyset\}$. Then,

$$A^c = \bigcap_{(p,q)\in A} [(p^c, \Sigma_2) \cup (\Sigma_1, q^c)] = (A_1^c, \Sigma_2) \cup (\Sigma_1, A_2^c) \bigcup_{S\in\mathscr{P}^*(A)} (S_1^c, (A\backslash S)_2^c)$$

where $\mathscr{P}^*(A) = \mathscr{P}(A) \backslash \{A, \emptyset\}$ and $\mathscr{P}(A)$ is the set of subsets of A. So,

$$X = A^c \cap A^{\perp c} = \bigcup_{(p,q)\in A} \{(p^{\perp c} \cap A_1^c, q^{\perp c}) \cup (p^{\perp c}, q^{\perp c} \cap A_2^c)$$

$$\bigcup_{S\in\mathscr{P}^*(\mathrm{A})} (p^{\perp c} \cap S_1^c, q^{\perp c} \cap (A\backslash S)_2^c)\}$$

If, for example, there exists $p \in A_1$ such that $p^{\perp c} \cap A_1^c \neq \emptyset$, then $X \neq \emptyset$ and the proof is finished. In consequence, we will suppose that $\forall p \in A_1$ and $q \in A_2$, $p^{\perp c} \subset A_1$ and $q^{\perp c} \subset A_2$. This implies

$$A_1 = \bigcup_{p\in A_1} p \subset \bigcup_{p\in A_1} p^{\perp c} \subset A_1$$

$$\Rightarrow A_1 = \bigcup_{p\in A_1} p^{\perp c} = \left(\bigcap_{p\in A_1} p^{\perp}\right)^c = \left(\bigcup_{p\in A_1} p\right)^{\perp c} = A_1^{\perp c}$$

and similarly for A_2. Since by hypothesis, $\mathscr{L}_1$ and $\mathscr{L}_2$ are irreducible, it follows that $A_1 = \Sigma_1$ and $A_2 = \Sigma_2$.

Let $p_0 \in A_1$. We write $b_{p_0} := \{q \in \Sigma_2 | (p_0, q) \in A\}$ (Fig. 1). We can suppose that $b_{p_0} \neq \Sigma_2$ because $A \neq \Sigma_1 \times \Sigma_2$. There exists $q_0 \in b_{p_0}$ such that $q_0^{\perp c} \not\subset b_{p_0}$. Indeed, suppose that $q^{\perp c} \subset b_{p_0}$ for every $q \in b_{p_0}$. Then,

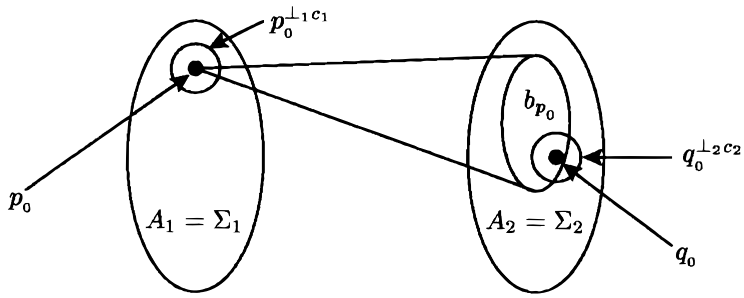


Fig. 1. Proof of Theorem 1, part (a).

$$b_{p_0} = \bigcup_{q \in b_{p_0}} q \subset \bigcup_{q \in b_{p_0}} q^{\perp c} \subset b_{p_0} \qquad \text{so} \quad b_{p_0} = \left(\bigcup_{q \in b_{p_0}} q \right)^{\perp c} = b_{p_0}^{\perp c}$$

which is impossible because $\mathcal{L}_2$ is irreducible. Let $S := A \backslash (p_0, b_{p_0})$; then $p_0 \in p_0^{\perp c} \cap S_1^c$ and $(A \backslash S)_2 = b_{p_0}$; so $q_0^{\perp c} \cap (A \backslash S)_2^c = q_0^{\perp c} \cap b_{p_0}^c \neq \emptyset$, which shows that $X \neq \emptyset$.

The remainder of the proof is an easy consequence. Let p be an atom of $\mathcal{L} := \oslash_{i \leq n} \mathcal{L}_i$. Write $z(p)$: $i \to \mathcal{Z}(p_i)$. We have to show that $\mathcal{Z}(p) = z(p)$.

We first verify that $z(p) \in \mathcal{Z}(\mathcal{L})$. From Lemma 1(a), $z(p) \in \mathcal{L}$ and by definition of $\perp_{\oslash}$,

$$z(p)^{\perp} = \bigcup_{i \leq n} o(\mathcal{Z}(p_i)^{\perp}) = \bigcup_{i \leq n} o(\mathcal{Z}(p_i)^c) = \Sigma \backslash z(p)$$

Finally, since $p < \mathcal{Z}(p) \wedge z(p)$, we have that $\mathcal{Z}(p) < z(p)$, and so, since $[O, z(p)] = \oslash_{i \leq n} [O, \mathcal{Z}(p_i)]$, $\mathcal{Z}(p) = z(p)$.

(b) The only thing we need to prove is that if $\mathcal{L}_\alpha$ are irreducible, then the separated product $\mathcal{L} := \oslash_{\alpha \in \omega} \mathcal{L}_\alpha$ is also irreducible. For the remainder it is easy to check that the argument of part (a) applies also for an arbitrary family.

Let τ be the topology on Σ defined in Lemma 2: Since $\mathcal{L}_\alpha$ are irreducible, by Lemma 2, Σ is a connected set of τ.

To conclude, it remains to remark that in a cao lattice, if (Σ, τ) is connected, then $\mathcal{L}$ is irreducible since every element of $\mathcal{Z}(\mathcal{L})$ is clopen. ∎

The remainder of this section is devoted to some examples of separated products.

Example 1. Let us illustrate Theorem 1 by a trivial example: Let $\mathcal{L}$ be a cao lattice. We first remark that $\mathcal{L} \simeq \natural_O^I \oslash \mathcal{L}$. So, by Theorem 1, $\mathcal{P}(\Omega) \oslash \mathcal{L}$ is orthoisomorphic to the Cartesian product $\prod_{\Omega} \mathcal{L}$.

Example 2. Consider $\mathcal{L} = P(\mathbb{C}^2) \circledwedge P(\mathbb{C}^2)$, where $P(\mathbb{C}^2)$ is the lattice of subspaces of $\mathbb{C}^2$. We denote by Σ the set of rays of $\mathbb{C}^2$; $I = \Sigma \times \Sigma$; p_1, q_1, r_1, p_2, q_2, r_2, x, and y elements of Σ; and $p = (p_1, p_2)$, $q = (q_1, q_2)$, and $r = (r_1, r_2,)$ are three atoms of $\mathcal{L}$. An easy calculation gives [see Lemma 1(b)]

$$p \vee q = \begin{cases} p \cup q & \text{if} \quad p_1 \neq q_1 \text{ and } p_2 \neq q_2 \\ (p_1, \Sigma) & \text{if} \quad p_1 = q_1 \text{ and } p_2 \neq q_2 \\ (\Sigma, p_2) & \text{if} \quad p_1 \neq q_1 \text{ and } p_2 = q_2 \end{cases}$$

$$r \vee (p_1, \Sigma) = \begin{cases} (p_1, \Sigma) & \text{if} \quad r_1 = p_1 \\ (p_1^{\perp}, r_2^{\perp})^{\perp} & \text{otherwise} \end{cases}$$

$$r \vee (p \cup q) = \begin{cases} I & \text{if} \quad p_1 \neq r_1 \neq q_1 \text{ and } p_2 \neq r_2 \neq q_2 \\ (p_1^{\perp}, q_2^{\perp})^{\perp} & \text{if} \quad r_1 \neq q_1 \text{ and } r_2 = q_2 \\ (q_1^{\perp}, p_2^{\perp})^{\perp} & \text{if} \quad r_1 \neq p_1 \text{ and } r_2 = p_2 \\ & \qquad \dots \end{cases}$$

So, apart from 0, I, and the atoms, $\mathcal{L}$ contains four types of elements: $(p_1, p_2) \cup (q_1, q_2)$ with $p_1 \neq q_1$ and $p_2 \neq q_2$, (p_1, Σ), (Σ, p_2), and finally $(p_1, p_2)^{\perp} = (p_1^{\perp}, \Sigma) \cup (\Sigma, p_2^{\perp})$. This is summarized in Fig. 2.

Moreover, $\mathcal{L}$ neither has the covering property nor it is orthomodular. It is a consequence of the Aerts theorem and it can be verified explicitly. For example, let us put $a := (p_1, p_2) \cup (q_1, q_2)$ with $p_1 \neq q_1$, $p_2 \neq q_2$, $p_1 \neq q_1^{\perp}$, and $p_2 \neq q_2^{\perp}$, $b := (p_1, \Sigma) \cup (\Sigma, q_2)$, and $r = (r_1, r_2)$ with $p_1 \neq r_1 \neq q_1$ and $p_2 \neq r_2 \neq q_2$. Then, $a^{\perp} = (p_1^{\perp}, q_2^{\perp}) \cup (q_1^{\perp}, p_2^{\perp})$, so $a^{\perp} \wedge b = O$ and in consequence, $a \vee (a^{\perp} \wedge b) = a \neq b$. Moreover, $r \vee a = I$, which does not cover a.

Example 3. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces over $\mathbb{C}$ and $P(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be the lattice of closed subspaces of the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$. We denote the orthogonality relation in $\mathcal{H}_1 \otimes \mathcal{H}_2$ induced by the scalar product by $\perp_{\otimes}$, Let μ: $P(\mathcal{H}_1) \circledwedge P(\mathcal{H}_2) \to P(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be defined by

$$\mu(a) := \overline{\langle \{ p \otimes q;\, (p, q) < a \} \rangle} = a^{\perp_{\otimes} \perp_{\otimes}}$$

Let Σ be the set of product vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and A a subset of Σ. Since $A^{\perp_{\circledwedge}} = A^{\perp_{\otimes}} \cap \Sigma$, it is easy to check that μ is an injective order-preserving map. Indeed, if $\mu(a) = \mu(b)$, then $a^{\perp_{\otimes}} = b^{\perp_{\otimes}}$ and so $a^{\perp_{\circledwedge}} = b^{\perp_{\circledwedge}}$, that is, $a = b$ since a and b are biorthogonal.

Moreover, the image of μ contains the set C of all closed subspaces V such that V and $V^{\perp_{\otimes}}$ are spanned by product vectors, and on $\mu^{-1}(C)$, μ preserves also the orthocomplementation. Indeed, define $A = V \cap \Sigma$ and

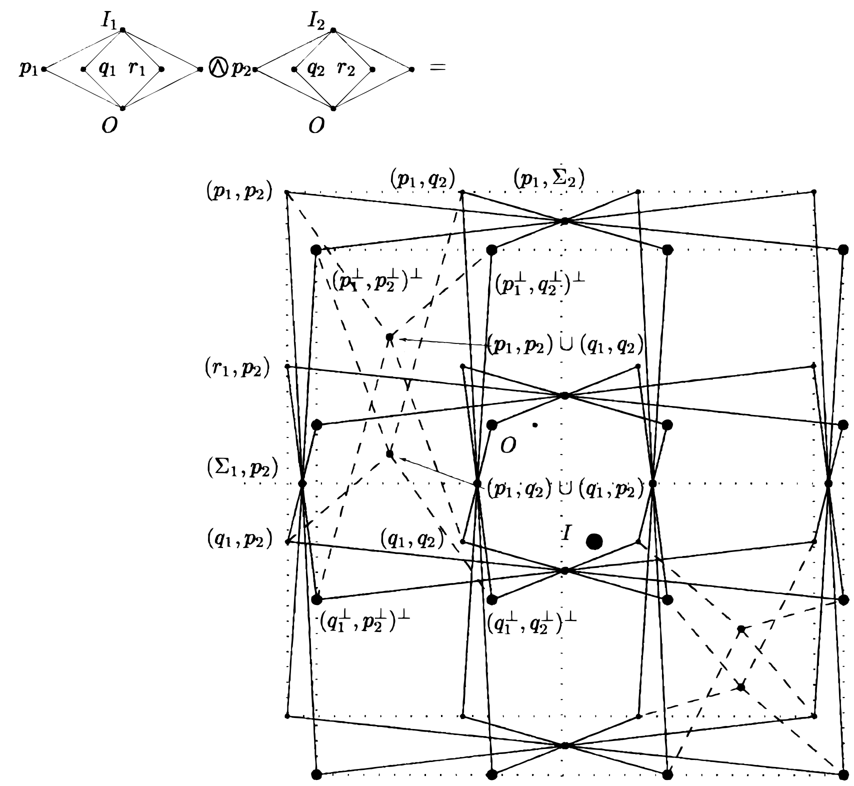


Fig. 2. The lattice as seen from above. The structure in dashed lines repeats for every box (α, β), (α, γ), (δ, β) (δ, γ), $(\alpha^\perp, \beta^\perp)^\perp$, $(\alpha^\perp, \gamma^\perp)^\perp$, ...

$B = V^{\perp\otimes} \cap \Sigma$. Then $A^{\perp\oslash} \subset V^{\perp\otimes}$ and so $A^{\perp\oslash} \subset B$ and similarly $B^{\perp\oslash} \subset A$. Moreover, $B \subset A^{\perp\oslash}$ and $A \subset B^{\perp\oslash}$. As a consequence, A and B are biorthogonal and $A^{\perp\oslash} = B$.

In Example 2, we can prove that $\dim(\mu(x^{\perp\oslash})) = \dim(\mu(x)^{\perp\otimes})$, $\forall x \in P(\mathbb{C}^2) \oslash P(\mathbb{C}^2)$, that is, that $\mathrm{Im}(\mu) = C$. So, since $P(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is orthomodular and $P(\mathbb{C}^2) \oslash P(\mathbb{C}^2)$ is not, the application μ preserves neither the meet nor the join. On can verify this explicitly: Remark that $\mu(b) = \langle\{ p_1 \otimes q_2, p_1 \otimes p_2, p_1^\perp \otimes q_2\}\rangle$. To see that $\mu(b) \wedge \mu(a^{\perp\oslash}) \neq \mu(b \wedge a^{\perp\oslash})$, it suffices to show that $\mu(b) \wedge \mu(a)^{\perp\otimes} \neq O$, in other words, that there exist two numbers α and β such that

$$\langle p_1 \otimes p_2, \alpha p_1 \otimes q_2 + \beta p_1 \otimes p_2 + p_1^\perp \otimes q_2\rangle = 0$$

$$\langle q_1 \otimes q_2, \alpha p_1 \otimes q_2 + \beta p_1 \otimes p_2 + p_1^\perp \otimes q_2\rangle = 0$$

that is, such that

$$\begin{pmatrix} \|p_1\|^2\langle p_2, q_2\rangle & \|p_1\|^2\|p_2\|^2 \\ \|q_2\|^2\langle q_1, p_1\rangle & \langle q_1, p_1\rangle\langle q_2, p_2\rangle \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} 0 \\ \langle q_1, p_1^{\perp}\rangle\|q_2\|^2 \end{pmatrix}$$

Indeed, the determinant $\|p_1\|^2 \langle q_1, p_1\rangle[|\langle p_2, q_2\rangle|^2 - \|p_2\|^2\|q_2\|^2]$ is different from 0 since $p_2 \neq q_2$ and $q_1 \neq p_1^{\perp}$.

Example 4. Finally, we give some examples of cao lattices which are not the separated product of two sublattices. First, by the Aerts theorem and Theorem 1, irreducible cao lattices which are orthomodular or have the covering property are never orthoisomorphic to the separated product of nontrivial cao lattices. But there are also irreducible cao lattices which are not orthomodular, which do not have the covering property, and are not isomorphic to the separated product of nontrivial cao lattices. Indeed, let $\mathscr{C}(S, \perp_S)$ be the lattice of biorthogonal subsets of an infinite-dimensional, noncomplete pre-Hilbertian space over $\mathbb{C}$ and let $\mathscr{L}$ be the cao lattice of Fig. 3. In the separated product of nontrivial cao lattices there always exist two elements a and b different from O and I such that $a \nless b$, $b \nless a$, and $a \vee b = a \cup b$ [Lemma 1(b)]. This is clearly not the case in $\mathscr{L}$.

3. JOIN-PRESERVING MAPS

This section is devoted to join-preserving maps between separated products. We recall that we call morphism a join-preserving map between two cao lattices, sending atoms to atoms and O to O, defined on a segment $D = [O, M]$ called the domain.

We first give a characterization of morphisms that we will use frequently: Let $\mathscr{L}_0$ and $\mathscr{L}_1$ be two cao lattices and f a morphism from $\mathscr{L}_0$ to $\mathscr{L}_1$ with domain $D = [O, M]$. Let g: $M \subset \Sigma_0 \to \Sigma_1$ be the restriction of f to the atoms of D. Then, the inverse image by g of a biorthogonal subset of Σ_1 is a biorthogonal subset of Σ_0; more precisely, $(g^{-1}(A^{\perp}))^{\perp\perp} = g^{-1}(A^{\perp})$ for any subset $A \subset \Sigma_1$. Moreover, $f(a) = g(a)^{\perp\perp}$, $\forall a < M$. Indeed, let p be an atom

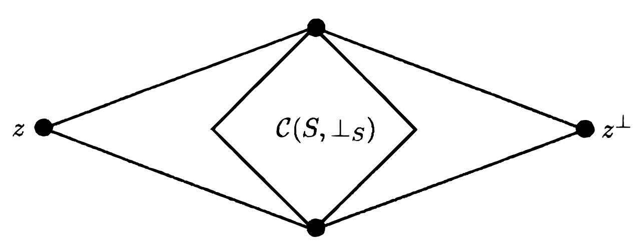


Fig. 3. Example 4.

of $\mathcal{L}_0$ such that $p \in (g^{-1}(A^{\perp}))^{\perp\perp}$; then p is in the domain of f and since f preserves the join, we have that $g(p) = f(p) < \vee_{q \in g^{-1}(A^{\perp})} f(q) < A^{\perp}$.

Reciprocally, let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two cao lattices, $M \in \mathcal{L}_0$ and g: $M \to \Sigma_1$ such that the inverse image by g of any biorthogonal subset is a biorthogonal subset; then f: $[O, M] \to \mathcal{L}_1$ defined by $f(a) := g(a)^{\perp\perp}$ is a morphism. First, $f(O) = O$. Let $\{a_\beta\}_{\beta \in \Omega}$ be a family of elements of $[O, M]$; then by definition, $\vee_{\beta \in \Omega} f(a_\beta) < f(\vee_{\beta \in \Omega} a_\beta)$. Moreover, since the inverse image by g of a biorthogonal subset is a biorthogonal subset, $\vee_{\beta \in \Omega} a_\beta < g^{-1}(\vee_{\beta \in \Omega} f(a_\beta))$ and so $g(\vee_{\beta \in \Omega} a_\beta) \subset \vee_{\beta \in \Omega} f(a_\beta)$ and $f(\vee_{\beta \in \Omega} a_\beta) < \vee_{\beta \in \Omega} f(a_\beta)$.

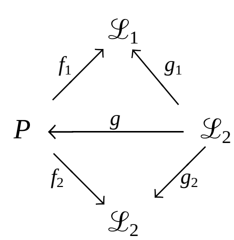
Finally, we remark that a bijective morphism f: $D \to \mathcal{L}_1$ is an isomorphism from D to $\mathcal{L}_1$, that is, a bijection which preserves the join and the meet. Indeed, since f is injective and preserves the join,

$$\begin{aligned} f(a) < f(b) &\Leftrightarrow f(a) \vee f(b) = f(b) \\ &\Leftrightarrow f(a \vee b) = f(b) \\ &\Leftrightarrow a \vee b = b \Leftrightarrow a < b \end{aligned}$$

which shows that $f(a) < f(b)$ implies $a < b$. In consequence, if q is an atom such that $q = f(x) < f(a) \wedge f(b)$, then $x < a$ and $x < b$, so $q < f(a \wedge b)$. Moreover, since f preserves the order, we have that $f(a \wedge b) < f(a) \wedge f(b)$.

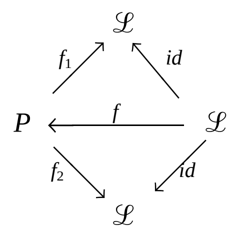
We remark that our definition of morphism differs from that in Moore (1995), where morphisms are partially defined maps g from $\Sigma_0 \backslash \mathcal{K}_0$ to Σ_1 such that $\mathcal{K}_0 \cup g^{-1}(A^{\perp\perp})$ is a biorthogonal for all $A \subset \Sigma_1$. Following Moore (1995), we call Prop′ the category of cao lattices with our definition of morphisms. Then, it is easy to check that the coproduct in Prop′ is the Cartesian product of cao lattices $(\Pi_\alpha \mathcal{L}_\alpha, f_\alpha)$ with $(f_\alpha(a))_\alpha = a$ and $(f_\alpha(a))_\beta = O$ for $\beta \neq \alpha$. But the separated product is not the product in this category (if indeed there is one).

This can be seen in two steps. First, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two elements of Prop′ and suppose that there exists a product P. Denote f_1 and f_2 the projection maps and h: $\Sigma_P \to \Sigma_1 \times \Sigma_2$ the map defined by $h(p) = (f_1(p), f_2(p))$. Then h is injective since the diagram



must admit a unique factorization, where g_i is the morphism sending I to $f_i(p)$. Further, h is surjective since for any atom $(p_1, p_2) \in \Sigma_1 \times \Sigma_2$ the preceding diagram must admit a factorization when g_i are the morphisms sending I to p_i.

Second, let $p \neq q$ be two atoms of a cao lattice $\mathscr{L}$. Let f be the morphism such that the following diagram commutes:



Then $f_1(f(p)) = p$, $f_2(f(p)) = p$, $f_1(f(q)) = q$, and $f_2(f(q)) = q$. Suppose that there exists an atom z different from p and q under $p \vee q$. Then $f(p) \neq f(z) \neq f(q)$ and since f preserves the join, $f(z) < f(p) \vee f(q)$, which shows by Lemma 1(b) that $P \neq \mathscr{L} \oslash \mathscr{L}$ (take for example $\mathscr{L} = \mathscr{L}_4$).

We now return to the purpose of the paper. For this section, we also need a preliminary lemma:

Lemma 3. Consider a family $\{\mathscr{L}_{0\alpha}\}_{\alpha\in\omega} \cup \mathscr{L}_1$ of cao lattices and a morphism f from $\mathscr{L}_0 := \oslash_{\alpha\in\omega} \mathscr{L}_{0\alpha}$ into $\mathscr{L}_1$. Let $\alpha \in \omega$. Suppose that $\mathscr{L}_{0\alpha}$ has the covering property. Let p and q be atoms of $\mathscr{L}_0$ in the domain of f such that $p_\beta = q_\beta$, $\forall\beta \neq \alpha$, and such that p_α and q_α have the same central cover. Denote respectively by x and y the images by f of p and q.

(a) If $x \neq y$, then $x \vee y \neq x \cup y$.

(b) Suppose, moreover, that $\mathscr{L}_1 = \oslash_{\alpha\in\omega'} \mathscr{L}_{1\alpha}$, where $\mathscr{L}_{1\alpha}$ are cao lattices. Then there exists at most one $\alpha \in \omega'$ such that $x_\alpha \neq y_\alpha$.

Proof. (a) Suppose that $x \vee y = x \cup y$. Since $\mathscr{L}_{0\alpha}$ has the covering property and p_α and q_α have the same central cover, there exist an atom r_α, different from p_α and q_α, such that $r_\alpha < p_\alpha \vee q_\alpha$. Let r be the atom of $\mathscr{L}_0$ defined by $r_\beta = p_\beta$, $\forall\beta \neq \alpha$. So, by Lemma 1(b), r is in the domain of f and since f preserves the join, $f(r) = x$ or $f(r) = y$. Suppose, for example, that $f(r) = x$. Then, since by the covering property and Lemma 1(b), $p \vee q = p \vee r$, we have

$$x \cup y = x \vee y = f(p) \vee f(q) = f(p \vee q) = f(p \vee r) = f(p) \vee f(r) = x$$

which shows that $x = y$.

(b) It is an immediate consequence of part (a) and Lemma 1(b). ■

When the evolution of a physical system can be described by an endomorphism f of the property lattice $\mathscr{L}$, the fact that f preserves the irreducible components of $\mathscr{L}$ means that the evolution of classical variables does not depend on the quantum variables. When $\mathscr{L}$ has the covering property, this is easily verified since two atoms have the same central cover if and only if $p \vee q \neq p \cup q$.

It is clear that in the general case, morphisms do not necesseraly preserve irreducible components, as we can see in the following example:

Example 5. Let $\mathscr{L} = \mathscr{C}(\mathbb{Z}_n, \perp_n)$, where $k^{\perp_n} := \{k + 2, \ldots, k + n - 2\}$ for $n \geq 4$ and $k^{\perp_n} := \{k + 1, \ldots, k + n - 1\}$ for $2 \leq n \leq 3$. It is easy to verify that $\perp_n$ is an orthogonality relation. Further, let g: $\mathbb{Z}_5 \to \mathbb{Z}_2$ be the map sending 0 to 0 and 1 to 1. It easy to check that the inverse image by g of any biorthogonal subset is biorthogonal, that is, that g defines a morphism f: $\mathscr{L}_5 \to \mathscr{L}_2$ with domain $M = \{0, 1\}$. Remark that this is the smallest example since, as is easy to prove, the smallest irreducible cao lattice is $\mathscr{L}_4$. Moreover, there are only two irreducible cao lattices with five atoms: $\mathscr{L}_5$ and the same lattice as in Fig. 3, but with $\mathscr{L}_3$ instead of $(S, \perp_S)$. The smallest example with a morphism with maximal domain, that is, with $D = [O, I]$ is f: $\mathscr{L}_6 \to \mathscr{L}_2$ with f sending atoms 0, 1, and 2 on atom 0 and atoms 3, 4, and 5 on atom 1.

In our case, that is, when $\mathscr{L} = \oslash_{\alpha \in \omega} \mathscr{L}_{0\alpha}$, endomorphisms preserve irreducible components if, among others, the following hypothesis on the domain $D = [O, M]$ is verified:

$\mathbf{H_{M1}}$ Atoms p and q in D having the same central cover are connected, that is:

Definition 2. Let $\{\mathscr{L}_\alpha\}_{\alpha \in \omega}$ be a family of cao lattices, $D = [O, M]$ be a segment of $\oslash_{\alpha \in \omega} \mathscr{L}_\alpha$ and p and q be two atoms of D having the same central cover.

(a) If ω is finite, we say that p and q are connected if there exists a finite set of atoms $r^0, \cdots r^n$ in $D \cap \mathscr{L}(p)$ such that $r^0 = p$, $r^n = q$ and for any $k < n$ there exists at most one j such that $r_j^k \neq r_j^{k+1}$.

(b) If ω is not finite, we say that p and q are connected if $C(p, q) \in D$.

Let us give an example: Suppose that $\#\omega = 2$; then by Theorem 1(a), p and q are connected, for example, if $(q_1, p_2) \in D$. Remark that Hypothesis H_{M1} is trivially satisfied if the domain of the morphism is maximal, that is, if $D = \mathscr{L}_0$.

For the proof of part (b) of Theorem 2, we develop a new general argument in the category <u>Prop′</u> involving the topology defined in Lemma 2.

Theorem 2. Consider a family $\{\mathcal{L}_{0\alpha}\}_{\alpha\in\omega} \cup \mathcal{L}_1$ of cao lattices and a morphism f from $\mathcal{L}_0 := \oslash_{\alpha\in\omega} \mathcal{L}_{0\alpha}$ into $\mathcal{L}_1$. Suppose that Hypothesis H_{M1} is satisfied and that $\mathcal{L}_{0\alpha}$ has the covering property $\forall\alpha$.

(a) If ω is finite, then for every atom p of D we have that $f(\mathcal{Z}(p) \wedge M) < \mathcal{Z}(f(p))$.

(b) If for any $\alpha \in \omega$, the join of every pair of atoms of $\mathcal{L}_{0\alpha}$ having the same central cover contains an infinity of atoms, then the result of part (a) is true for an arbitrary family.

Proof. (a) Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ be two atoms in the domain of f having the same central cover. First, by Theorem 1(a) we know that $\mathcal{Z}(p_i) = \mathcal{Z}(q_i)$, $\forall_i \leq n$. We write $x = f(p)$ and $y = f(q)$.

If there exists only one j such that $p_j \neq q_j$, then the theorem is a direct consequence of Lemma 3(a) since if $\mathcal{Z}(x) \neq \mathcal{Z}(y)$, then $x \vee y = x \cup y$.

Otherwise, consider the atoms defined in Hypothesis H_{M1}. Then by hypothesis, $\mathcal{Z}(p) = \mathcal{Z}(r^1) = \cdots = \mathcal{Z}(r^{n-1}) = \mathcal{Z}(q)$ and we have that

$$\mathcal{Z}(f(p)) = \mathcal{Z}(f(r^1)), \mathcal{Z}(f(r^1)) = \mathcal{Z}(f(r^2)), \ldots$$
$$\mathcal{Z}(f(r^{n-1})) = \mathcal{Z}(f(q))$$

(b) Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two cao lattices and f be a morphism from $\mathcal{L}_0$ into $\mathcal{L}_1$ with domain $D = [O, M]$ and g its restriction to the atoms of D.

As we have seen, the inverse image by g of any biorthogonal subset of Σ_1 is a biorthogonal subset. In consequence, g is a continuous application from (M, τ_0) into (Σ_1, τ_1), where τ_i is the topology defined in Lemma 2. For this topology, elements of $\mathcal{Z}(\mathcal{L}_1)$ are clopen and, thus, connected sets are contained in atoms of $\mathcal{Z}(\mathcal{L}_1)$. Since the image by a continuous application of a connected set is a connected set, to prove the theorem it suffices, to show that if two atoms p and q of D have the same central cover, there exists a connected set $\mathcal{C}$ containing them and contained in the domain M.

For our case, by Lemma 2 and Hypothesis H_{M1}, we can put $\mathcal{C} = C(p, q)$. ■

Remark 2. The argument used in case (b) works of course also for a finite family of cao lattices. By hypothesis H_{M1} and lemma 1(d), it suffices to put $\mathcal{C} \vee_{k=0\cdots n} r^k = \cup_{k=0\cdots n} z^k$. Indeed, $\mathcal{C} < M$ and $\mathcal{C}$ is the union of connected sets z_k with $z_k \cap z_{k+1} \neq \emptyset$, that is, $\mathcal{C}$ is a connected set. Remark, that if ω is infinite countable, first $\mathcal{C}$ is also connected but does not contains q if $q_k = p_k$ only for a finite set of k, and second that if $q \wedge \mathcal{C} = O$, $\mathcal{C} \cup q$ is not connected.

Corollary 1. Suppose moreover that $\mathcal{L}_1 = \oslash_{\alpha\in\omega} \mathcal{L}_{1\alpha}$, where $\mathcal{L}_{1\alpha}$ are cao lattices having the covering property and that f is bijective.

(a) In both cases of Theorem 2, if $\mathcal{L}_1$ satisfies the same hypotheses as $\mathcal{L}_0$ and if the domain of f is maximal, that is, if $D = \mathcal{L}_0$, then $f(\mathfrak{X}(p)) = \mathfrak{X}(f(p))$ for every atom p of $\mathcal{L}_0$.

(b) If for every $\alpha \in \omega$ the join of any pair of atoms of $\mathcal{L}_{k\alpha}$ (where $k =$ 0 or 1) having the same central cover contains an infinity of atoms, then $f(\mathfrak{X}(p) \wedge M) = \mathfrak{X}(f(p))$ for every atom p of D.

Proof. Part (a) is easy. For part (b), suppose that $f(\mathfrak{X}(p) \wedge M) \neq \mathfrak{X}(f(p))$. Since f is surjective, there exists $a \in D$ such that $f(a) = \mathfrak{X}(f(p))$. Then, $a = \coprod_{\gamma \in \Omega_0} a \wedge \gamma$ and in consequence, since f is injective, $\mathfrak{X}(f(p)) = \coprod_{\gamma \in \Omega_0} f(a \wedge \gamma)$, which is impossible, because by Lemma 2, $\mathfrak{X}(f(p))$ is a connected set. ■

We now turn to Theorem 3. We need some more notations: Let $\{\mathcal{L}_{0i}\}_{i \in I}$ and $\{\mathcal{L}_{1i}\}_{i \in I}$ be two finite families of cao lattices and f a morphism from $\mathcal{L}_0 := \oslash_{i \in I} \mathcal{L}_{0i}$ into $\mathcal{L}_1 := \oslash_{i \in I} \mathcal{L}_{1i}$ with domain $D = [O, M]$. For an atom p of D, $k \in I$, and r_k an atom of $\mathcal{L}_k$, we write

$$x^k(p) := \bigwedge_{i \neq k} o(p_i) \wedge M$$

$$M_k(p) := P_k(x^k(p)), \qquad M_k := \bigcup_{q \in D} M_k(q) \tag{5}$$

where P_i: $\Pi_{i \in I} \Sigma_{0l} \rightarrow \Sigma_{0l}$ is the natural projection and q are atoms. We remark that $M_k(p)$ is a nonempty biorthogonal subset [Lemma 1(a)], but that in general M_k is not biorthogonal. We will need one more hypotheses on the domain of f. For $k \in I$ we have:

$\mathbf{H^k_{M2}}$ For atoms $p \neq q$ in D such that $p_i \neq q_i$ only for one $i \in I$, $X_k := M_k(p) \wedge M_k(q) \neq O$, and if $f(x^k(p))$ and $f(x^k(q))$ are not atoms, then $f(y^k(p))$ and $f(y^k(q))$ are not atoms, where, for example $y^k_k(p) = X_k$ and $y^k_i = p_i$ for $i \neq k$.

We remark that this hypothesis is also trivially satisfied for any $k \in I$ if the domain is maximal, since in this case $M_k(p) = \Sigma_{0k}$. Moreover, if $\#I = 2$ anf if $\mathcal{L}_{0i}$ are irreducible, it easy to see that this hypothesis implies that atoms in D are connected.

Theorem 3. Let $\{\mathcal{L}_{0\alpha}\}_{\alpha \in \omega}$ and $\{\mathcal{L}_{1\alpha}\}_{\alpha \in \omega}$ be two families of irreducible cao lattices having the covering property, σ a bijection of ω, $\{f_\alpha\}_{\alpha \in \omega}$ a family of morphisms from $\mathcal{L}_{0\alpha}$ into $\mathcal{L}_{1\sigma(\alpha)}$ with domain $D_\alpha = [O, M_\alpha]$, and f a morphism from $\mathcal{L}_0 = \oslash_{\alpha \in \omega} \mathcal{L}_{0\alpha}$ into $\mathcal{L}_1 = \oslash_{\alpha \in \omega} \mathcal{L}_{1\alpha}$ with domain $D = [O, M]$.

(a) The application $\Pi_{\alpha\in\omega} f_\alpha\colon \mathscr{L}_0 \to \sigma(\mathscr{L}_1) := \oslash_{\alpha\in\omega} \mathscr{L}_{1\sigma(\alpha)}$ defined by

$$\left(\prod_{\alpha\in\omega} f_\alpha\right)(a) := \bigvee_{p<a} \{f_\alpha(p_\alpha)\}_{\alpha\in\omega}$$

where p are atoms, is a morphism, bijective if and only if f_α are bijective.

(b) Suppose that ω is finite and write $I = \omega$. If the Hypotheses $\mathrm{H}^k_{\mathrm{M2}}$ and H_{M1} are satisfied for every $k \in I$, and if $f(M)$ is not trivial in the sense that for any $i \in I$ and for any atom $p_i \in \mathscr{L}_{1i}$, $f(M)$ is not contained in $o(p_i)$, then there exist a bijection σ of I and a family of join-preserving maps sending atoms to atoms $\{f_i\}_{i\in I}$ from $\mathscr{L}_{0i}$ into $\mathscr{L}_{1\sigma(i)}$ defined only on the subset M_i of atoms of Σ_{0i} defined in (5) (which are not necessary biorthogonal), such that $f = \sigma^{-1} \circ (\Pi_{i\in I} f_i)$.

As a consequence, if the subsets M_i are biorthogonal, then f_i are morphisms, so in particular if $D = \mathscr{L}_0$.

Proof. (a) We call g_i the restriction of f_i to atoms. For an atom p of $\mathscr{L}_0$, we write $g(p) := \{g_i(p_i)\}_{i\in I}$. Now, $g(p)$ is an atom of $\sigma(\mathscr{L}_1)$ and as we remarked at the beginning of this section, it suffices to show that the inverse image by g of any biorthogonal subset a of $\sigma(\Sigma_1) := \Pi_{i\in I} \Sigma_{i\sigma(i)}$ is a biorthogonal subset of Σ_0:

$$g^{-1}(a) = g^{-1}\left(\bigcap_{p<a^\perp} p^\perp\right) = \bigcap_{p<a^\perp}\left(g^{-1}\left(\bigcup_{i\in I} o(p^\perp_{\sigma(i)})\right)\right)$$
$$= \bigcap_{p<a^\perp}\left[\bigcup_{i\in I} g^{-1}(o(p^\perp_{\sigma(i)}))\right]$$

where p are atoms and by definition of $o(\cdots)$,

$$g^{-1}(o(p^\perp_{\sigma(i)}))_i = g_i^{-1}(p^\perp_{\sigma(i)}), \qquad g^{-1}(o(p^\perp_{\sigma(i)}))_j = g_j^{-1}(\Sigma_{1\sigma(j)})$$

Since f^i is a morphism and $g_i^{-1}(p^\perp_{\sigma(i)}) < g_i^{-1}(\Sigma_{1\sigma(i)}) = M_i$, by Lemma 1(c), the elements between brackets are biorthogonal subsets.

Finally, we only verify that if f_i are bijective, then $f := \Pi_{i\in I} f_i$ is also bijective, the other implication is easy to show.

First f is bijective on the atoms, so $b = f(g^{-1}(b))$ for every $b \in \sigma(\mathscr{L}_0)$, and thus f is surjective.

To show that f is injective, it suffices to see that $g^{-1}(f(a)) = a$, that is, that $g^{-1}(f(a)) < a$, for every $a \in \mathscr{L}_0$. By (4), we know that

$$a^{\perp\perp} = \bigcap_{h\in\omega^a} \bigcup_{\alpha\in\mathrm{Im}(h)} o(h^{-1}(\alpha)^{\perp\perp})$$

Moreover,

$$f(a) = \left(\bigcap_{p<a} \bigcup_{\alpha\in\omega} o(f_\alpha(p_\alpha)^\perp)\right)^\perp = \left(\bigcup_{h\in\omega^a} \bigcap_{\alpha\in \mathrm{Im}(h)} o(g_\alpha(h^{-1}(\alpha))^\perp)\right)^\perp$$
$$= \bigcap_{h\in\omega^a} \bigcup_{\alpha\in \mathrm{Im}(h)} o(f_\alpha(h^{-1}(\alpha)^{\perp\perp}))$$

since f_α preserves the join.

As a result, if $g(p) < f(a)$, then for all $h \in \omega^a$ there exist $\alpha \in \omega$ such that $g_\alpha(p_\alpha) \in f_\alpha(h^{-1}(\alpha)^{\perp\perp})$, which shows, since f_α is a bijective morphism, that $p_\alpha \in h^{-1}(\alpha)^{\perp\perp}$, that is, that $g(p) < f(a)$ implies $p < a$.

(b) The proof of this part is divided into four steps:

Step 1. Let p be an atom of $\mathscr{L}_0$ in the domain D (Fig. 4.). Then for any $k \in I$, there exist $j_k(p) \in I$ such that $P_l(f(x^k(p)))$ is an atom for all $l \neq j_k(p)$ [for notations, see (5)]. Indeed, let q_k be an atom of $\mathscr{L}_{0k}$ in $M_k(p)$. Let q be the atom such that $q_i = p_i$, $\forall_i \neq k$. Suppose that $f(q) \neq f(p)$. By Lemma 3(b) we know that there exists only one $j \in I$ such that $P_j(f(q)) \neq P_j(f(p))$. Denote this j by $j_k(p)$. Suppose now that there exist an atom r_k in $M_k(p)$ and $l \neq j_k(p)$ such that $P_l(f(r)) \neq P_l(f(p))$, where r is the atom such that $r_i = p_i$, $\forall i \neq k$. So by Lemma 3(b), $P_{j_k(p)}(f(r)) = P_{j_k(p)}(f(p))$ and since $P_{j_k(p)}(f(q)) \neq P_{j_k(p)}(f(p))$, $P_l(f(q)) = P_l(f(p))$, and so $P_l(f(r)) \neq P_l(f(q))$ and $P_{j_k(p)}(f(r)) \neq P_{j_k(p)}(f(q))$, which is impossible because of Lemma 3(b).

Step 2. Next, for any $k \in I$, we show that there exists $j \in I$ such that for any $p \in D$ and any $l \neq j$, $P_l(f(x^k(p)))$ is an atom of $\mathscr{L}_{1l}$, in other words that for all $k \in I$, one can always choose the function $j_k(\cdot)$ such that $j_k(p) = j_k(q)$ $\forall p, q \in D$ (Fig. 5).

Let $l \in J$ and p and q be atoms in D such that $p_i = q_i$, $\forall_i \neq l$ and $p_l \neq q_l$, and let $k \neq l$. Suppose that $f(x^k(p))$ and $f(x^k(q))$ are not atoms and that $j_k(p) \neq j_k(q)$ [for notations, see (5)].

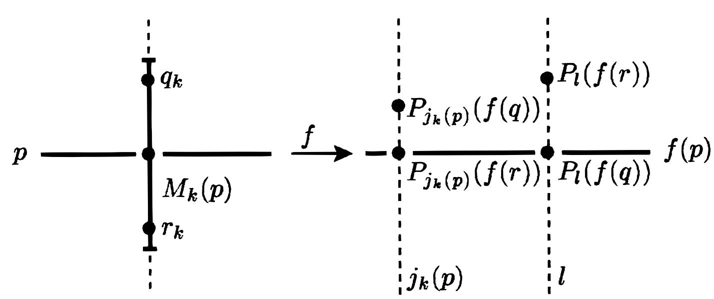


Fig. 4. Proof of Theorem 3(b), Step 1.

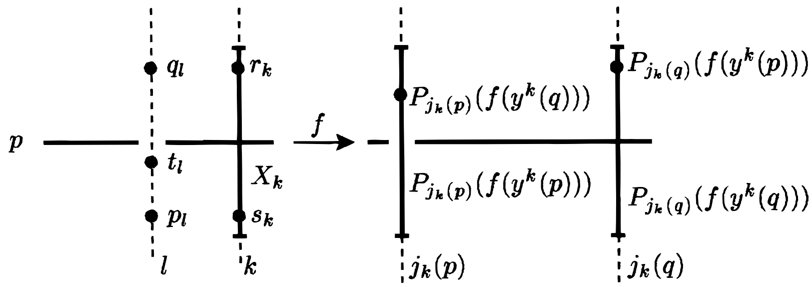


Fig. 5. Proof of Theorem 3(b), Step 2.

Then, we have that

$$P_{jk(p)}(f(y^k(q))) < P_{jk(p)}(f(y^k(p))) \tag{6}$$

$$P_{jk(q)}(f(y^k(p))) < P_{jk(q)}(f(y^k(q)))$$

(for notations, see Hypothesis $\mathrm{H}^k_{\mathrm{M}_2}$). Indeed, suppose, for example, that

$$P_{jk(p)}(f(y^k(q))) \wedge P_{jk(p)}(f(y^k(p))) = O \tag{7}$$

By Hypothesis $\mathrm{H}^k_{\mathrm{M}_2}$, there exist two atoms $r_k \neq s_k \in X_k := M_k(p) \wedge M_k(q)$ such that $f(s^q) \neq f(r^q)$, where $r^q_i = s^q_i = q_i$, $\forall i \neq k$, $r^q_k = r_k$, and $s^q_k = s_k$. But, (7) implies that $P_{jk(q)}(f(r^q)) = P_{jk(q)}(f(r^p))$ and $P_{jk(q)}(f(s^q)) = P_{jk(q)}(f(s^p))$. As a consequence, since $P_{jk(q)}(f(y^k(p)))$ is an atom, $(f(q^p) = f(r^q)$, which is a contradiction.

By (6) and Lemma 1(b), since f preserves the join, we have that

$$f(y^k(p) \vee y^k(q)) = f(y^k(p)) \cup f(y^k(q)) \tag{8}$$

since for any m different from $j_k(q)$ and $j_k(p)$, $P_m(f(y^k(p)) = P_m(f(y^k(q)))$. But since $\mathscr{L}_{0l}$ has the covering property, there exist an atom t_l different from p_l and q_l such that $t_l < p_l \vee q_l$. Define $y^k(t)$ by $y^k(t)_l = t_l$ and $y^k(t)_j = y^k(p)_j$ for $j \neq l$. Then $y^k(t) < y^k(p) \vee y^k(q)$. So $y^k(t)$ is in D and by (8), $f(y^k(t)) < f(y^k(p)) \cup f(y^k(q))$. Since there exists only one $m \in I$ such that $P_m(f(y^k(t)))$ is not an atom, $f(y^k(t)) < f(y^k(p))$ or $f(y^k(t)) < f(y^k(q))$. Suppose that $f(y^k(t)) < f(y^k(p))$. Then, by the covering property and by Lemma 1(b) we have that $y^k(t) \vee y^k(p) = y^k(p) \vee y^k(q)$, that is, since f preserves the join, $f(y^k(q)) < f(y^k(p))$, which is impossible because of Hypothesis $\mathrm{H}^k_{\mathrm{M}_2}$; $f(y^k(q))$ is not an atom.

We have shown that for all $k \in I$, $j_k(p) = j_k(q)$, $\forall p, q \in D$, such that $p_i = q_i$, $\forall i \neq j$, for a given $j \in I$. Moreover, the same result for arbitrary atoms p and q in D follows since by hypothesis they are connected. As a consequence, we can define j_k as $j_k(p)$ for an arbitrary atom p in D.

Step 3. We now show that if $k \neq l$, then $j_k \neq j_l$, in other words, since is I is finite, that σ: $k \to j_k$ is a permutation of I.

Let $k \neq l$, and suppose that $j_k = j_l$; then $I \backslash \sigma(I) \neq \emptyset$. Let $m \in I \backslash \sigma(I)$. Then $P_m \circ f$ is constant, which contradicts the hypothesis on the image of f. Indeed, let p and q be atoms in D and $i \in I$ such that $p_k = q_k$, $\forall k \neq i$. Then, since $j_i \neq m$, by Step 1, $P_m(f(p)) = P_m(f(q))$.

Step 4. Let $p_k \in M_k$ [for notation, see (5)]. Let q be an atom in D such that $q_k = p_k$. We define f_k as

$$f_k(p_k) := P_{\sigma(k)}(f(q)) \qquad \text{with} \quad q \in D \text{ and } q_k = p_k$$

We now have to check that this definition is independent of the choice of q. Let r be an atom of D such that $r_k = p_k$ and r_i different from q_i only for $i = j$. So q and r are in $x^j(q)$ and since we have shown in Step 3 that σ is a permutation, we know that $P_{\sigma(k)}\, f(x^j(q))$ is an atom, which shows that $P_{\sigma(k)}(f(q)) = P_{\sigma(k)}(f(r))$. ■

Remark 3. In the general case where the lattices $\mathcal{L}_{ij}$ are reducible, the applications f_i depend on the atoms of the center of $\mathcal{L}_0$ which are of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where α_i are atoms of $\mathcal{Z}(\mathcal{L}_{0i})$. But in general, f_i depend on α and not only on α_i. Let us give a simple example: Let $\mathcal{H}$ be a Hilbert space over the complex numbers $\mathbb{C}$, and $U_1 \neq U_2$ two unitary applications on $\mathcal{H}$. Put $\mathcal{L}_0 = \mathrm{P}(\mathcal{H}) \oslash (\mathrm{P}(\mathcal{H}) \times \mathrm{P}(\mathcal{H}))$, $\mathcal{L}_1 = \mathrm{P}(\mathcal{H}) \oslash \mathrm{P}(\mathcal{H})$, $f(p, q, O) = (U_1(p), U_2(q))$, and $f(p, O, q) = (U_2(p), U_1(q))$.

Remark 4. Suppose that for any $i \in I$ and for any atoms $p_i \neq q_i$ in M_i, there exist two atoms p and q in D such that $P_i(p) = p_i$, $P_i(q) = q_i$, and $p_k \not\perp q_k$ for any $k \neq i$. Suppose, moreover, that f satisfies $f(p) \perp f(q) \to p \perp q$ for any atoms p and $q \in D$. Then we have that $f_i(p_i) \perp f_i(q_i) \to p_i \perp q_i$ for any atoms p_i and $q_i \in M_i$. Indeed, if $f_i(p_i) \perp f_i(q_i)$, then $f(p) \perp f(q)$; thus, by the hypothesis on f, $p \perp q$, and by the hypothesis on the domain and definition of $\perp_{\oslash}$, $p_i \perp q_i$.

As a consequence, if, moreover, $\mathcal{L}_{0i}$ are Hilbertian lattices over the complex field, if M_i are biorthogonal (i.e., f_i are morphisms), and if the rank of $\mathrm{Im}(f_i)$ is greater than or equal to 3, then f_i are induced by semilinear maps which are unitary or antiunitary if they are quasilinear (see the introduction).

Finally, if the domain is maximal, then we have that the f_i preserve the orthogonality relation for all i (that is, $f_i(p_i) \perp f_i(q_i) \Leftrightarrow p_i \perp q_i$, $\forall\, p_i, q_i \in \mathcal{L}_{0i}$) if and only if $\Pi_{i \in I} f_i$ preserves the orthogonality relation $\perp_{\oslash}$.

4. CONCLUSION

We have shown that in a model where two separated entities with classical degrees of freedom are described by the separated product of two

Cartesian products over the classical degrees of freedom of Hilbertian lattices and the evolution is described by a morphism, the systems can only interact via their classical variables. We made two hypotheses on the domain of the morphism, H_{M1} and H_{M2}, which physically mean essentially that for a given initial state of the first system, there exist a large number of states of the second system for which the system does not disappear during the evolution.

In a forthcoming paper, we will show that the relation: $\Phi_{01}(\alpha) \in Q_{t_0}$ for every $\alpha \in Q_{t_1}$, is *a priori* not satisfied in the construction of Aerts of the property lattice of two separated entities when they interact. This work was supported in part by the Swiss National Science Foundation.

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